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Complementary conjecture revisited

Jie-Tai Yu*

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

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Abstract

By a known case of the Jacobian conjecture, we give a simple elementary proof of the two dimensional complementary conjecture. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [7], Moh, McKay and Wang proved the following.

Theorem 1. *Let $p_{11}(t), p_{12}(t), p_{21}(t), p_{22}(t)$ be polynomials with zero constant terms and let $A(X,Y), B(X,Y) \in k[X,Y]$ (k is a field of characteristic zero) be irreducible polynomials such that*

$$A(p_{11}(t), p_{12}(t)) = 0, \qquad B(p_{11}(t), p_{12}(t)) = t,$$
$$A(p_{21}(t), p_{22}(t)) = t, \qquad B(p_{21}(t), p_{22}(t)) = 0.$$

Then $k[A(X,Y), B(X,Y)] = k[X,Y]$. Namely, $(A(X,Y), B(X,Y))$ is a polynomial automorphism.

This theorem was called 'two variable face polynomials conjecture', or 'two dimensional complementary conjecture' in [9]. In fact in [9] the conjecture was formulated for arbitrary n. For $n > 2$, to our best knowledge, the conjecture is still open.

* E-mail: yujt@hkuxa.hku.hk.

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The proof of Theorem 1 in [7] uses notions of the characteristic data of two polynomials and approximate roots of a polynomial introduced in [1,2], as well as the proof of the famous Abhyankar–Moh Theorem (the theorem of embeddings of the line in the plane) in [3], which is very complicated.

In this note, we present an alternative proof for Theorem 1, by a known case of the Jacobian conjecture. The proof in this paper is much simpler than that in [7]. We feel that this proof gives some new insight of the problem, hence it is worthwhile to record the proof here, in the hope that the ideas used here will be useful for attacking the two-dimensional Jacobian conjecture and the n-dimensional complementary conjecture in the future.

2. Preliminaries

Lemma 2. *Let k be a field of characteristic zero and let $F=(F_1,F_2)\in(k[X,Y])^2$ be such that $\det(J(F))\in k^*$ and the Newton polygon of $F_1(G_1(X,Y),G_2(X,Y))$ is a triangle or a line segment for all two dimensional polynomial automorphisms $G=(G_1,G_2)\in(k[X,Y])^2$. Then F is an automorphism.*

Lemma 2 was first proved in [4, Theorem 19.4, p. 143] by very simple and elmentary methods, as a trivial case of the Jacobian conjecture which says $F\in(k[X_1,\dots,X_n])^n$ (k is a field of characteristic zero) is an automorphism if $\det(J(F))\in k^*$. To our best knowledge, the conjecture is still open for $n\geq 2$ (the case $n=1$ is trivially true). For a history and related topics on the Jacobian conjecture, see [5].

Recall that for $f(t),g(t)\in k[t]$ with $f(t)\notin k$, (k is a field), the minimal polynomial of $f(t)$ and $g(t)$ over k is the unique (up to a factor in k^*) irreducible polynomial $M(X,Y)\in k[X,Y]$ determined by $f(t)$ and $g(t)$. The minimal polynomial $M(X,Y)$ of $f(t)$ and $g(t)$ over k has the following very important property: if $L(X,Y)\in k[X,Y]$ with $L(f(t),g(t))=0$, then $M(X,Y)\mid L(X,Y)$ in $k[X,Y]$.

Lemma 3. *Let $f(t),g(t)\in k[t]$ (k is a field) with $k[f,g]=k[t]$ and let $M(X,Y)\in k[X,Y]$ be the minimal polynomial of $f(t)$ and $g(t)$ over k. Then*

$$\frac{f'(t)}{M_Y(f(t),g(t))}=-\frac{g'(t)}{M_X(f(t),g(t))}\in k^*.$$

Proof. It is a direct consequence of [6, Proposition 1.1]. □

Recall that for any polynomial $P(X,Y)\in k[X,Y]$ (k is a field), its Newton polygon is defined as the convex hull of $(0,0)$ in union with the lattice points (i,j) for which X^iY^j appears in $P(X,Y)$ with a non-zero coefficient.

Lemma 4. *Let k be a field and let $f(t),g(t)\in k[t]$ such that $f(t)\notin k$ and let $M(X,Y)$ be the minimal polynomial of $f(t)$ and $g(t)$ (that is, M is an irreducible polynomial* *in* $k[X,Y]$ *and* $M(f(t),g(t))=0$). *Then the Newton polygon of* $M(X,Y)$ *is a triangle or a line segment.*

Proof. By [8, Theorem 1, p. 246], $\mathrm{Res}_t(f(t)-X,g(t)-Y)=cM(X,Y)^q$ where $c\in k^*$ and q is a positive integer. By [8, Corollary 6, p. 250], the Newton polygon of $\mathrm{Res}_t(f(t)-X,g(t)-Y)$ is a triangle or line segment, so is that of $M(X,Y)$.

3. Proof of Theorem 1

By hypothesis, $A(X,Y)$ is the minimal polynomial of $p_{11}(t)$ and $p_{12}(t)$ over k, $B(X,Y)$ is the minimal polynomial of $p_{21}(t)$ and $p_{22}(t)$ over k. Since $A(p_{21}(t),p_{22}(t)) = t$, we have

$$p'_{21}(t)A_X(p_{21}(t),p_{22}(t))+p'_{22}(t)A_Y(p_{21}(t),p_{22}(t))=1.$$

By Lemma 3,

$$(B_YA_X-B_XA_Y)(p_{21}(t),p_{22}(t))=e_1\in k^*.$$

Hence

$$\det(J(A,B))(p_{21}(t),p_{22}(t))=e_1\in k^*.$$

Similarly,

$$\det(J(A,B))(p_{11}(t),p_{12}(t))=e_2\in k^*.$$

Substituting $t=0$, we get $e_1=e_2=(\det(J(A,B))(0,0)):=e$. Since $B(X,Y)$ is the minimal polynomial of $p_{21}(t)$ and $p_{22}(t)$, we get $B(X,Y)\mid\det(J(A,B))-e$. Similarly, $A(X,Y)\mid\det(J(A,B))-e$. By the hypothesis of Theorem 1, $A(X,Y)$ and $B(X,Y)$ are relatively prime in $k[X,Y]$. Hence

$$A(X,Y)B(X,Y)\mid\det(J(A,B))-e.$$

If $\det(J(A,B))-e\neq 0$, then $\deg(AB)\le\deg(\det(J(A,B))$ (here deg denotes the total degree with respect to both X and Y). But obviously we have

$$\deg(AB)=\deg(A)+\deg(B)>\deg(A)+\deg(B)-2\ge\deg(J(A,B)).$$

This contradiction shows that $\det(J(A,B))-e=0$. Therefore $\det(J(A,B))=e\in k^*$. Given any two dimensional polynomial automorphism $G=(G_1,G_2)$ with $G^{-1}=H=(H_1,H_2)$. Since $A(X,Y)$ is the minimal polynomial of $p_{11}(t)$ and $p_{12}(t)$, it is easy to see that $A(G_1(X,Y),G_2(X,Y))$ is the minimal polynomial of $H_1(p_{11}(t),p_{12}(t))$ and $H_2(p_{11}(t),\ p_{12}(t))$. By Lemma 4, the Newton polygon of $A(G_1(X,Y),G_2(X,Y))$ is a triangle or a line segment. By Lemma 2, (A,B) is an automorphism of $k[X,Y]$.

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